An Exact Expression for the Noise Resistance R_n for the Hawkins Bipolar Noise Model

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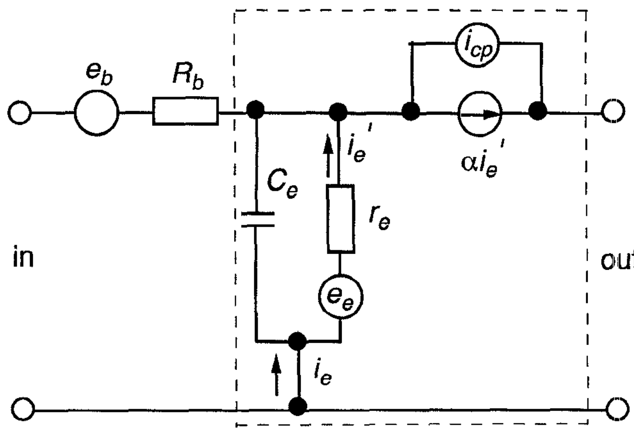
***Abstract*—A much used noise model for the bipolar transistor is that based on Hawkins formulation. The model, however lacks an expression for the noise resistance. An exact analytic expression, based on the noise correlation matrix approach, is developed for the noise resistance for this noise model.**

I. Introduction

PERHAPS the most used analytic formulation of the noise parameters of a bipolar transistor in the common emitter mode is that of Hawkins [1]. Hawkin's noise model, Fig. 1, is essentially a low-to-medium frequency model in that it neglects all device capacitances except the emitter junction capacitance, and all parasitic resistances except the base resistance, and does not include a phase shift factor in the current gain. In Fig. 1, we denote the intrinsic portion of the model enclosed with dashed lines. Three noise sources are present, the two intrinsic shot noise sources, e_e and i_{cp}, plus the thermal noise source of the extrinsic base resistance. These sources can be considered "white" for all practical purposes. All three sources are uncorrelated.[1] Their noise spectral power densities are indicated by the formulas associated with Fig. 1. The overbar implies a statistical average.

Hawkins derived simple analytic expressions for the minimum noise figure and the optimum source impedance. Lacking in this model is the expression for the noise resistance R_n. In this letter, we present a simple analytic expression for this noise parameter derived from the noise correlation matrix approach [2].

To apply the noise correlation matrix approach we transform the noise model in Fig. 1 to an equivalent one consisting of two noise sources, a voltage source and a current source preceding a noiseless version of the bipolar circuit of Fig. 1. The transformed noise model takes the form shown in Fig. 2. Since the system is linear, the two noise sources of Fig. 2 can be expressed in terms of the three original noise sources by a linear transformation. This also holds for noise power spectra as represented by the noise power correlation matrices [2]. These matrices are *defined,* respectively, for the intrinsic device as $\boldsymbol{N}$ and for the transformed noise circuit as $\boldsymbol{C}$, where



$$\overline{i_{cp}i_{cp}^*} = 2kT_o g_e(\alpha_o - |\alpha|^2)B \qquad \alpha = \frac{\alpha_o}{1 + j\frac{f}{f_b}} \qquad r_e = \frac{kT}{qI_e}$$

$$\overline{e_e e_e^*} = 2kT_o r_e B$$

$$\overline{i_{cp} e_e^*} = 0 \qquad \beta_o = \frac{\alpha_o}{1 - \alpha_o} \qquad g_e = r_e^{-1}$$

$$\overline{e_b e_b^*} = 4kT_o R_b B$$

Fig. 1. Hawkins noise model for the bipolar transistor.

$$\boldsymbol{N} = \frac{1}{4kT_oB}\begin{bmatrix} \overline{e_e e_e^*} & \overline{e_e i_{cp}^*} \\ \overline{i_{cp} e_c^*} & \overline{i_{cp} i_{cp}^*} \end{bmatrix} = \begin{bmatrix} \frac{r_e}{2} & 0 \\ 0 & \frac{\alpha_o - |\alpha|^2}{2r_e} \end{bmatrix} \tag{1a}$$

$$\boldsymbol{C} = \frac{1}{4kT_oB}\begin{bmatrix} \overline{e_n e_n^*} & \overline{e_n i_n^*} \\ \overline{i_n e_n^*} & \overline{i_n i_n^*} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}. \tag{1b}$$

Note that the noise sources in Fig. 2, generally speaking, are correlated ($C_{12} \neq 0$). Here, k denotes Boltzmann's constant, $T_o = 293$ K. The standard reference temperature, and B the incremental noise bandwidth. As noted before, the overbar denotes the statistical average, the asterisk the complex conjugate.[2] We wish to point out that the transformation from the circuit of Fig. 1 to Fig. 2 *does not involve any simplifications or approximations* in the formulation to follow.

The noise correlation matrix $\boldsymbol{C}$ can be obtained in terms of $\boldsymbol{N}$ by a straightforward application of the steps outlined by

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[1]Note that although the emitter and collector *physical* noise sources, in reality, are correlated in a bipolar transistor, Hawkins transformed these sources to a special *circuit* configuration (Fig. 1) that cancels the correlation of the *equivalent* circuit sources. See [1] for details.

[2]We represent the noise spectra over the positive frequency range only, unlike [2], which uses the doubly-infinite frequency range. This choice is arbitrary and merely replaces the factor $2kTB$ by $4kTB$. This factor "cancels out" in the determination of the four terminal noise parameters and therefore does not affect the end results. In addition, we retain the bandwidth designation $\boldsymbol{B}$ rather than setting it to unity.

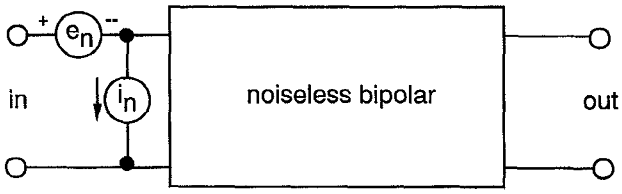

Fig. 2. The transformed noise model for the bipolar transistor used.

Hillbrand and Russer [2]. Thus we find

$$C = AZTN(AZT)\dagger + ARA\dagger, \quad (2)$$

where the dagger denotes the Hermitian conjugate. The matrix Z is just the inverse of the admittance matrix Y for the intrinsic portion of Hawkin's model, and T is a transformation matrix which converts Hawkins' intrinsic noise sources e_e and i_{cp} to shunt current sources, respectively, across the base-emitter and collector-emitter ports of the intrinsic transistor. Thus,

$$Y = \begin{bmatrix} (1-\alpha)g_e + j\omega C_e + Y_c & -Y_c \\ \alpha g_e - Y_c & Y_c \end{bmatrix}$$

$$T = \begin{bmatrix} -\frac{1-\alpha}{r_e} & 1 \\ -\frac{\alpha}{r_e} & -1 \end{bmatrix}, \quad (3)$$

where $g_c = 1/r_e$. We have added a fictitious admittance Y_c across the α-current generator in Fig. 1 to overcome, temporarily, the singularity of the actual Z matrix. However, in the final evaluation of (2), this admittance is set equal to zero, as its effect in this limit "cancels" out. The matrices A and R are given by

$$A = \begin{bmatrix} 1 & -\frac{z_{11}+R_b}{z_{21}} \\ 0 & -\frac{1}{z_{21}} \end{bmatrix}$$

$$R = \frac{1}{4kT_oB}\begin{bmatrix} \overline{e_b e_b^*} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R_b & 0 \\ 0 & 0 \end{bmatrix}, \quad (4)$$

where the $z_{i,j}$ denote the matrix elements of Z. Note that A is a circuit transformation matrix, whereas R is a noise correlation matrix representing the thermal noise of the extrinsic base resistance. Thus, the thermal noise of the base resistance manifests itself as an additive term to the intrinsic noise of the bipolar, signified by the first term in (2).

The elements of the noise correlation matrix C in (1b) contain all necessary information about the four extrinsic noise parameters $F_{\min}, R_{s,\text{opt}}, X_{s,\text{opt}}$, and R_n of the bipolar. Hawkins [1] has given simple analytic expressions for the first three. We shall present the missing expression for R_n. This is especially easy to extract from the matrix C since $R_n = C_{11}$.[3] By a straightforward but lengthy expansion of C_{11} as derived from (2) in the limit $Y_c \to 0$, we obtain the

[3] This follows from the definition $R_n = \overline{e_n e_n^*}/4kT_oB$.

TABLE I
COMPARISON OF THE VALUES OF R_n FOR THE EXAMPLE TREATED BY HAWKINS CALCULATED BY USE OF (5) AND FORMULAE IN [3] AND [4]

F(GHz)	(5)	[3]	[4]
1	14.89	15.20	14.58
2	15.35	16.60	14.91
3	16.13	18.93	15.45
5	18.68	26.40	17.20
7	22.71	37.61	19.81
10	32.06	61.42	25.37

following expression

$$R_n = R_b\left(D - \frac{1}{\beta_0}\right) + \frac{r_e}{2}\left[D + \left(\frac{R_b}{r_e}\right)^2 \cdot \left(1 - \alpha_0 + \left(\frac{f}{f_b}\right)^2 + \left(\frac{f}{f_e}\right)^2 + \left(\frac{1}{\beta_0} - \left(\frac{f}{f_b}\right)\left(\frac{f}{f_e}\right)\right)^2\right)\right], \quad (5)$$

where f_b denotes the "cutoff" frequency of the common base current gain $\alpha(f)$, Fig. 1, as obtained by fitting the equivalent circuit to measurements, and f_c represents the emitter time constant

$$f_e = \frac{1}{2\pi r_e C_e}. \quad (6)$$

The dummy variable D is given by

$$D = \frac{1 + \left(\frac{f}{f_b}\right)^2}{\alpha_o^2}. \quad (7)$$

All remaining parameters are defined in Fig. 1. We must point out that since no approximations were made in the derivation of the correlation matrix expression (2) or in the evaluation of the C_{11} matrix element, the expression for R_n is exact *for the noise model of Hawkins.* Note, that (5) applies to the unpackaged bipolar (chip). Package parasitics may have a profound effect on the noise presistance measured at the package terminals.

We have applied (5) to the example treated by Hawkins, namely:

$$I_e = 4 \text{ mA}$$
$$I_c \approx \alpha_0 I_e$$
$$\alpha_0 = 0.98(\beta_0 = 50)$$
$$C_e = 2.65 \text{ pF}$$
$$r_e = 6.5\,\Omega$$
$$R_b = 11\,\Omega$$
$$f_b = 23 \text{ GHz}$$
$$f_e = 9.24 \text{ GHz}$$
$$f_r = 4.42 \text{ GHz}.$$

For comparison, we have also applied these parameter values to two alternative, but approximate expressions for R_n one

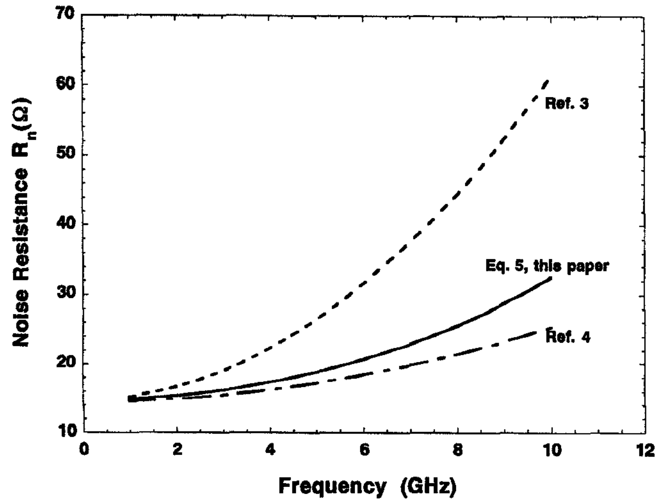

Fig. 3. Comparison of various analytic formulae for R_n.

proposed by Fukui [3], and another, a modification of Fukui's expression by Vendelin *et al.* [4].

The values (in ohms) derived by use of the three formulas are listed in Table I and are plotted in Fig. 3. Notice that Fukui's values (third column) increase too rapidly with frequency. On the other hand, the simplification of Fukui's formula (fourth column) exhibits too mild a frequency dependence. Unfortunately, Hawkins did not cite any experimental values for R_n for his example and we were unable to obtain any noise resistance data for this chip device from vendor's catalog.

II. Summary

A simple, but exact formula for R_n has been presented for the Hawkins noise model of the bipolar transistor. This formula can be used in conjunction with the Hawkins formula for $F_{\min}, R_{s,\mathrm{opt}}$, and $X_{s,\mathrm{opt}}$ to provide a complete set of equations for representing the *low-medium* frequency range noise performance of a bipolar transistor in chip form. We caution the reader that the range of validity of the derived expression for R_n should be confined to that of the Hawkins model, itself. In practice, the neglected equivalent circuit parameters in the Hawkins model and the effect of embedding the chip in a package must be taken into account at higher frequencies.

References

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